

Distribution of instanton sizes in a simplified instanton gas model

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Abstract. We investigate the distribution of instanton sizes in the framework of a simplified model for ensembles of instantons. This model takes into account the non-diluteness of instantons. The infrared problem for the integration over instanton sizes is dealt with in a self-consistent manner by approximating instanton interactions by a repulsive hard core potential. This leads to a dynamical suppression of large instantons. The characteristic features of the instanton size distribution are studied by means of analytic and Monte Carlo methods. In one dimension exact results can be derived. In any dimension we find a power law behaviour for small sizes, consistent with the semi-classical results. At large instanton sizes the distribution decays exponentially. The results are compared with those from lattice simulations.

1 Introduction

1.1 Instantons in gauge theories

In non-abelian gauge theories different topologically non-trivial configurations have been made responsible for non-perturbative features [1–5]. An important class are instantons, which are solutions of the Euclidean field equations with non-vanishing topological charge [6]. They give contributions to the saddle-point approximation of Euclidean functional integrals, which lead to non-perturbative effects [7,8]. For a review see [10].

In the dilute gas approximation [8] one considers superpositions of single instantons as quasi saddle points of the action. These configurations are characterized by the instanton positions $\{\mathbf{a}_j\}$ in four-dimensional space-time, the instanton sizes $\{\rho_j\}$, and other internal parameters. For a single instanton the action is

$$S_1 = \frac{8\pi^2}{g_0^2}, \quad (1)$$

where g_0 is the bare gauge coupling constant. Taking into account quadratic fluctuations around the one-instanton solution [7,8] its contribution to the functional integral is

$$Z_1 = \int d^4\mathbf{a} \int d\rho C\rho^{-5} \exp\left\{-\frac{8\pi^2}{g^2(1/\rho)}\right\}, \quad (2)$$

where $g(\mu)$ is the running coupling. In the case of supersymmetric Yang-Mills theory this formula has even been established at the two-loop-level, in ordinary Yang-Mills theory there are higher-order corrections to the integrand [9]. In one-loop order the running coupling obeys

$$g^2(\mu) = \frac{8\pi^2}{b \log(\mu/\Lambda)} \quad (3)$$

where

$$b = \frac{11N}{3}, \quad (4)$$

and Λ is the renormalization-group invariant scale parameter, so that

$$Z_1 = \int d^4\mathbf{a} \int d\rho C\rho^{-5}(\rho\Lambda)^b. \quad (5)$$

Therefore the integrand is proportional to ρ^{b-5} and increases with increasing instanton size.

The space-time integral over the instanton position gives the usual volume factor, which is needed in the large volume limit in order to get an extensive free energy. The integral over ρ , however, represents an infrared problem. Where the instanton density becomes important, for $\rho \geq O(1/\Lambda)$, we leave the region of validity of the semi-classical expansion because the running coupling becomes too large.

It should be noted that the apparent infrared divergence in (5) is an artifact of using the one-loop formula for $g^2(1/\rho)$ long after it has become invalid, i.e. for $\rho \geq 1/\Lambda$. Nevertheless we are confronted with the infrared problem in the size integration for the single instanton contribution Z_1 . If the semiclassical approximation is meaningful at all, a solution of this problem in the context of the full instanton ensemble is required.

The quasi saddle points composed of any number of instantons and anti-instantons are treated as independent in the dilute gas approximation. Consequently their contributions exponentiate in the usual way. The problem with the integration over the sizes persists and has to be dealt with. The simplest way is to cut the integrations off at

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some ad-hoc value ρ_c . But since the integrand increases with increasing ρ_j the dominant contribution comes from large ρ_j near the cut-off where the assumption of diluteness fails. Moreover the introduction of an ad-hoc cut-off leads to inconsistencies with the renormalization group [11].

In order to solve the problem of the instanton size integrations it has been proposed that instanton sizes are cut off in a dynamical way [11,12]. The dynamical cut-off should originate from configurations where instantons start to overlap. Configurations of overlapping instantons have an action which deviates from the sum of the single instanton actions. Therefore large instantons feel an interaction. Additionally, the fluctuations around the multi-instanton configurations contribute to the instanton interaction. The interaction between instantons is expected to suppress overlapping instantons and to result in a dynamical self-consistent cut-off.

Some consequences of this picture have been discussed in [11,12], based on certain assumptions about the repulsive instanton interactions. In [12] a temporary infrared cut-off was introduced by means of a finite space-time volume V . The large V limit was then considered with the help of renormalization group arguments. This led to some general results independent of the specific form of the repulsive instanton interactions. In particular, a finite renormalization factor

$$\frac{4}{b} = \frac{12}{11N} \quad (6)$$

appears in some quantities, e.g. in the trace anomaly [11], correcting inconsistencies with the renormalization group. The same factor is conjectured to multiply the instanton singularities in the Borel plane, which then coincide with the infrared renormalons. The conclusions were supported by considering a model of the instanton ensemble, where the repulsive interaction is approximated by a hard core.

The theory of instanton ensembles with a dynamical size cut-off has been developed further by Shuryak [13], see [10] for a review. In his model of an “instanton liquid” various observables have been calculated and related to hadronic phenomenology.

In connection with the dynamical cut-off the distribution of instanton sizes is of central importance. The size-distribution reflects the way in which large instantons are suppressed and thus gives information about the instanton interactions. In recent years it has been studied by means of lattice Monte Carlo calculations by different groups [14–17]. For small sizes the distribution is predicted to be

$$n(\rho) \sim \rho^{b-5} \quad (7)$$

by the dilute gas approximation as well as by the “instanton liquid model”, in accordance with (5). For large sizes ρ , where the dynamical cut-off is in effect, not much is known about the distribution. There are arguments [11, 18, 19] in favour of a suppression like

$$n(\rho) \sim \exp(-c\rho^p) \quad \text{with } p = 2. \quad (8)$$

In this article we investigate the distribution of instanton sizes in a model [12] where the instanton interactions

are approximated by a repulsive hard core of variable size. Although this approximation appears to be crude, the general features of the instanton ensemble with a dynamical cut-off are present. In particular, using analytical and numerical methods we calculate the asymptotic behaviour for small and for large sizes ρ and compare them with results from Monte Carlo simulations of lattice gauge theory. More details can be found in [20].

In [12] it has been conjectured that the distribution $n(\rho)$ is affected by the finite renormalization factor $4/b$ in such a way that for small ρ asymptotically

$$n(\rho) \sim \rho^{-1} \rho^{\frac{4}{b}(b-4)}. \quad (9)$$

Using the simplified model, we shall show below that this conjecture is wrong and that instead the semiclassical result (7) holds.

1.2 Simplified model for ensembles of instantons

Consider an ensemble of instantons in d space-time dimensions. In the spirit of [12] we introduce a finite volume V and study the approach to the thermodynamic limit $V \rightarrow \infty$.

In the sector with instanton number K the partition function is written as

$$Z_K(V) = \frac{C^K}{K!} \int \prod_{i=1}^K d\mathbf{a}_i d\rho_i \prod_{j=1}^K \rho_j^{b-d-1} \times e^{-U(\{\mathbf{a}_k\}, \{\rho_k\})}, \quad (10)$$

where the instanton positions and radii are denoted $\{\mathbf{a}_i, \rho_i\}$. C is a constant, whose numerical value is unimportant here, $b = 11N/3$ for $SU(N)$ Yang-Mills theory, and U represents the interaction potential between instantons. Distances are measured in units of Λ^{-1} .

In our simplified model the repulsive potential is approximated by a hard core potential. The radius of an instanton core varies proportional to the size ρ of the instanton. In a finite volume V this means

$$e^{-U(\{\mathbf{a}_i\}, \{\rho_i\})} = \Theta(\{\mathbf{a}_i\}, \{\rho_i\}) \quad (11)$$

with

$$\Theta(\{\mathbf{a}_i\}, \{\rho_i\}) = 1, \quad \text{if } \begin{cases} 1. & \|\mathbf{a}_i - \mathbf{a}_j\| > \left(\frac{\tau}{v_1}\right)^{\frac{1}{d}} (\rho_i + \rho_j) \quad \forall i, j, \text{ and} \\ 2. & |a_i^\mu| < \frac{1}{2} V^{\frac{1}{d}} \quad \forall \mu, i, \text{ and} \\ 3. & 0 < \rho_i < \frac{1}{2} \left(\frac{v_1}{\tau} V\right)^{\frac{1}{d}} \quad \forall i \end{cases}$$

$$\Theta(\{\mathbf{a}_i\}, \{\rho_i\}) = 0, \quad \text{else.}$$

Here

$$v_1 = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)} \quad (12)$$

is the volume of the unit sphere in d dimensions. The parameter τ specifies the effective volume $\tau\rho_j^d$ of an instanton and is of the order of v_1 .

We introduce a reduced distribution by

$$Z_K^{red}(V, \rho_K) = \frac{C^K}{K!} \int \prod_{i=1}^K da_i \prod_{j=1}^{K-1} d\rho_j \times \prod_{k=1}^K \rho_k^{b-d-1} \Theta(\{a_i\}, \{\rho_l\}), \quad (13)$$

such that

$$Z_K(V) = \int d\rho Z_K^{red}(V, \rho). \quad (14)$$

For the total system with variable instanton number the grand canonical partition function is

$$Z(V) = \sum_{K=0}^{\infty} Z_K(V), \quad \text{where } Z_0(V) = 1. \quad (15)$$

We do not distinguish between instantons and anti-instantons in this model. In this way we neglect aspects of the interactions which differ between instantons and anti-instantons, but we do not expect that they play a significant role for our considerations.

The probability distribution of instanton numbers is given by

$$\mathcal{P}_K(V) = \frac{Z_K(V)}{Z(V)}. \quad (16)$$

In order to define the probability distribution of instanton sizes one has to specify how the sizes are sampled. The definition should be made in such a way that it is compatible with the Monte Carlo calculations to be discussed later. In the Monte Carlo runs configurations with a variable number of instantons are produced. A configuration of K instantons contributes K entries to the total histogram of instanton sizes. Therefore it has a relative weight proportional to K . Correspondingly the probability distribution in the K -instanton sector is normalized to K [21]:

$$\bar{n}_K(V, \rho) = K \frac{Z_K^{red}(V, \rho)}{Z_K(V)}. \quad (17)$$

In the total ensemble the sizes are then distributed according to

$$\bar{n}(V, \rho) = \sum_{K=1}^{\infty} \mathcal{P}_K(V) \bar{n}_K(V, \rho) \quad (18)$$

$$= \frac{1}{Z(V)} \sum_{K=1}^{\infty} K Z_K^{red}(V, \rho) \quad (19)$$

with

$$\int_0^{\infty} \bar{n}(V, \rho) d\rho = \langle K \rangle_V. \quad (20)$$

As the expectation value of the instanton number grows linearly with the volume V one is interested in the rescaled distribution

$$n(V, \rho) = \frac{\bar{n}(V, \rho)}{V}. \quad (21)$$

In the following sections we study the properties of $n(V, \rho)$, and its thermodynamic limit $n(\rho)$, respectively, utilizing analytical approaches as well as Monte Carlo-methods.

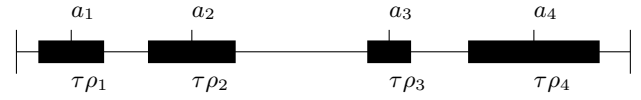


Fig. 1. One-dimensional instanton gas, $K = 4$

2 The one-dimensional instanton gas

In $d = 1$ dimensions the model can be solved exactly in the thermodynamic limit. This case illustrates some general features and can serve as a testing ground for approximations used in higher dimensions. Therefore we shall discuss these results before we turn to the consideration of other dimensions.

In the one-dimensional case the canonical partition function for a system of spatial length L can be written as

$$Z_K(L) = \frac{C^K}{K!} \int \prod_{i=1}^K da_i d\rho_i \prod_{j=1}^K \rho_j^{b-2} \Theta(\{a_k\}, \{\rho_k\}) \quad (22)$$

with

$$\Theta(\{a_i\}, \{\rho_i\}) = 1, \quad \text{if } \begin{cases} 1.) & |a_i - a_j| > (\frac{\tau}{2}) (\rho_i + \rho_j) \quad \forall i, j, \text{ and} \\ 2.) & |a_i| < \frac{1}{2}L \quad \forall i, \text{ and} \\ 3.) & 0 < \rho_i < \frac{L}{\tau} \quad \forall i \end{cases}$$

$$\Theta(\{a_i\}, \{\rho_i\}) = 0, \quad \text{else}$$

This represents a system of rods with variable lengths on a line. A pictorial representation is given in Fig. 1.

Integration over the instanton positions $\{a_i\}$ yields the effective free volume of K indistinguishable particles on the line L :

$$\int \prod_{i=1}^K da_i \Theta(\{a_k\}, \{\rho_k\}) = \bar{\Theta}(\{\rho_k\}) \left(L - \tau \sum_{j=1}^K \rho_j \right)^K, \quad (23)$$

with

$$\bar{\Theta}(\{\rho_k\}) = \begin{cases} 1, & \text{if } \tau \sum_{j=1}^K \rho_j \leq L \\ 0, & \text{else,} \end{cases} \quad (24)$$

as can be shown by induction. With this result the partition function reads

$$Z_K(L) = \frac{C^K}{K!} \int_0^{\frac{L}{\tau}} d\rho_K \cdots \int_0^{\frac{L}{\tau} - \sum_{j=2}^K \rho_j} d\rho_1 \times \prod_{j=1}^K \rho_j^{b-2} \left(L - \tau \sum_{l=1}^K \rho_l \right)^K. \quad (25)$$

The integrations over the ρ_j can be carried out successively employing

$$\int_0^1 x^a (1-x)^b dx = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}, \quad (26)$$

and one obtains for the size distribution

$$n(L, \rho) = \left\{ \sum_{K=1}^{\infty} KC\gamma^{K-1} (\Gamma(b(K-1)+2))^{-1} \rho^{b-2} \times (L - \tau\rho)^{b(K-1)+1} \right\} / \left\{ \sum_{K=0}^{\infty} \gamma^K (\Gamma(bK+1))^{-1} L^{bK+1} \right\} \quad (27)$$

with

$$\gamma = \frac{C\Gamma(b-1)}{\tau^{b-1}}. \quad (28)$$

The asymptotic behaviour for small instanton sizes is given by the power law

$$n(L, \rho) \sim \text{const.} \rho^{b-2} \quad \text{for } \rho \rightarrow 0 \quad (29)$$

in the sense of

$$b-2 = \lim_{\rho \rightarrow 0} \frac{\ln(n(L, \rho))}{\ln(\rho)}. \quad (30)$$

In order to discuss the behaviour for large ρ we take the infinite volume limit

$$n(L, \rho) \xrightarrow{L \rightarrow \infty} n(\rho). \quad (31)$$

The grand canonical sums can be evaluated by replacing them by integrals over K and performing a saddle point approximation, which becomes exact in the large- L limit. The result is

$$n(\rho) = \frac{C}{b} \rho^{b-2} e^{-c\rho}, \quad (32)$$

with

$$c = (C\Gamma(b-2)\tau)^{\frac{1}{b}}. \quad (33)$$

In addition to the power law with exponent $b-2 = b-d-1$ we recognize an exponential suppression of large instanton sizes. The exponent p in the exponential, cp. (8), is equal to 1 in one dimension. Our next aim is to see how these results generalize to higher dimensions d .

3 The general instanton gas

In higher dimensions, $d > 1$, it is not possible to derive closed expressions for the partition functions or the size distributions. The main difficulty is that the integrations over the instanton positions and the radii cannot be decoupled. In particular we have to use approximations for the effective free volume of sets of instantons in higher dimensions. Nevertheless one can obtain approximate expressions for $n(V, \rho)$ and derive its asymptotic behaviour for small ρ .

The canonical K -instanton partition function $Z_K(V)$ is written as

$$Z_K(V) = \frac{C^K}{K!} \int_0^{\bar{V}} dv \int \prod_{i=1}^K d\mathbf{a}_i d\rho_i \prod_{j=1}^K \rho_j^\alpha \times \Theta(\{\mathbf{a}_k\}, \{\rho_k\}) \delta(v - \tau \sum_l \rho_l^d), \quad (34)$$

where

$$\alpha = b - d - 1, \quad (35)$$

and

$$v = \tau \sum_l \rho_l^d \leq \bar{V} \quad (36)$$

is the total effective volume of instantons. The maximal accessible volume $\bar{V} \leq V$ accounts for the fact that spheres in dimensions $d > 1$ cannot completely occupy a given volume V .

An approximative decoupling of the integrations can be achieved through the observation that for given $v = \tau \sum_j \rho_j^d$ the product

$$\prod_{j=1}^K \rho_j^\alpha$$

develops a sharp maximum at

$$\rho_j = \left(\frac{v}{K\tau} \right)^{1/d} = \rho_0, \quad j \in \{1, \dots, K\} \quad (37)$$

in the thermodynamic limit. The main idea is then to perform a saddle point approximation for the ρ -integrations near this sharp maximum. The integrations over the positions \mathbf{a}_j then correspond to a gas of hard spheres with equal radii ρ_0 . With

$$\rho_j = \rho_0 + \delta_j \quad (38)$$

and

$$\delta \left(v - \tau \sum_{j=1}^K \rho_j^d \right) \approx \delta \left(-\tau d \rho_0^{d-1} \sum_{j=1}^K \delta_j \right) = \frac{\rho_0^{1-d}}{2\pi\tau d} \int dq e^{iq \sum_{j=1}^K \delta_j} \quad (39)$$

the ρ -integrations can be solved straightforwardly.

In a similar way the reduced partition function $Z_K^{red}(V, \rho)$ can be evaluated. For a given radius ρ of the K^{th} instanton one assigns an effective volume $\bar{V} - \tau\rho^d$ to the remaining $K-1$ instantons and performs the saddle point approximation for the integrations over $\rho_1, \dots, \rho_{K-1}$ in terms of Gaussian integrals.

The integral over the instanton positions

$$\frac{1}{K!} \int \prod_{j=1}^K d\mathbf{a}_j \Theta(\{\mathbf{a}_j\}, \{\rho_0\}) \quad (40)$$

can be estimated with the help of geometrical considerations [22, 21, 23, 12, 20]. We use an approximation of the form

$$\frac{1}{K!} \int \prod_{j=1}^K d\mathbf{a}_j \Theta(\{\mathbf{a}_j\}, \{\rho_0\}) \approx \frac{1}{K!} (V - v_{eff})^K, \quad (41)$$

where

$$v_{eff} = h(d)v, \quad (42)$$

and $h(d) = V/\bar{V}$ measures the inverse filling fraction of spheres in a volume V . In the one-dimensional case we get $h(1) = 1$ because a given length can be completely filled with rods. For $d > 1$ we consider $h(d)$ as a parameter. Lower and upper bounds are given by $1 \leq h(d) \leq 2^{d-1}$. For more details on this point see [22, 21, 23, 12, 20].

Using these expressions we get by some lengthy but straightforward calculations for the canonical partition functions

$$Z_K(V) \approx \sqrt{\frac{\alpha}{2\pi}} \frac{\sqrt{K}}{d} \left(\sqrt{\frac{2\pi}{\alpha}} \frac{C}{(K\tau h(d))^{\beta-1}} \right)^K \times \frac{\Gamma((\beta-1)K)}{\Gamma(\beta K + 1)} V^{\beta K}, \quad (43)$$

and for the reduced ones

$$Z_K^{\text{red}}(V, \rho) \approx \sqrt{\frac{\alpha}{2\pi}} \frac{\sqrt{K-1}}{d} C \rho^\alpha \left(\sqrt{\frac{2\pi}{\alpha}} \frac{C}{((K-1)\tau h(d))^{\beta-1}} \right)^{K-1} \times \frac{\Gamma((\beta-1)(K-1))}{\Gamma(\beta(K-1)+2)} \tilde{V}^{\beta(K-1)+1}, \quad (44)$$

where

$$\tilde{V} = V - h(d)\tau\rho^d, \quad \beta = \frac{b}{d}. \quad (45)$$

The next step is to perform the grand-canonical sums over the instanton numbers K . As in the one-dimensional case, the sums can be evaluated in the large volume limit by replacing them by integrals which are calculated by means of the saddle point method. The error of this approximation vanishes in the thermodynamic limit. For the size distribution we get in this way

$$n(\rho) = \frac{C^d}{b} \rho^{b-d-1} \exp(-c\rho^d), \quad (46)$$

with

$$c^{\frac{b}{d}} = C \sqrt{\frac{2\pi}{b-d-1}} \left(\frac{b}{d} - 1 \right)^{\frac{b}{d}-1} e^{-(\frac{b}{d}-1)} h(d) \tau. \quad (47)$$

For $b \gg d$ this takes the form

$$c^{\frac{b}{d}} = C \Gamma \left(\frac{b}{d} - 1 \right) d^{-\frac{1}{2}} h(d) \tau, \quad (48)$$

which agrees with the one-dimensional result.

The expression for $n(\rho)$ is consistent with the general expectation mentioned in the introduction: for small ρ it grows powerlike with an exponent $\alpha = b - d - 1$, and for large ρ this power-law is combined with an exponential decrease.

Although the canonical partition functions are dominated by configurations where the instantons are densely packed, the exponent α agrees with the one of the semi-classical dilute gas approximation. This result does not depend on the details of our approximations and follows

from the general structure of the occurring terms in the grand-canonical sums. The conjecture, made in [12], that due to the denseness of instantons the small- ρ behaviour of $n(\rho)$ gets modified, is therefore wrong.

On the other hand, the value of the exponent $p = d$ in the exponential decay at large ρ should be considered with reservations, because it depends on the saddle point approximations which have been made. In $d = 1$ dimensions it is correct, but we would not be surprised, if in higher dimensions the true value would differ from d . In order to get more insight into this question and to get an idea of the quality of the approximations being made so far, we have also studied the instanton gas by grand canonical Monte Carlo simulations.

4 Monte Carlo simulations

Usually Monte Carlo calculations are done in the canonical ensemble. In our case the particle number has to change and it is necessary to simulate a grand canonical ensemble. Simulations of grand canonical systems are not very common. They are rarely discussed in the literature and some important details remain unclear. Therefore it appears appropriate to describe the algorithm we have used in our calculations. For related work on this topic we refer to [24–29].

4.1 Grand canonical Monte Carlo algorithms

A stochastic process, which is realized in a Monte Carlo simulation, is specified by a transition matrix $W(X, Y)$, where X and Y denote states of the system. For the purpose of a simulation W is usually decomposed as a product of two factors: $\omega(X, Y)$ represents a proposal probability for a transition from X to Y , and a_{XY} denotes the corresponding acceptance probability. In addition to normalization and ergodicity one has to require stationarity, which is often fulfilled by demanding the stronger METROPOLIS condition of detailed balance:

$$a_{XY} = \min \left(1, \frac{\omega(Y, X)P(Y)}{\omega(X, Y)P(X)} \right). \quad (49)$$

Here P is the probability distribution, which we want to generate as the stationary distribution of the underlying stochastic process. In our context it is given by

$$P_K(V; \mathbf{a}_1, \dots, \mathbf{a}_K, \rho_1, \dots, \rho_K) = \frac{1}{Z(V)} \frac{C^K}{K!} \prod_{j=1}^K \rho_j^\alpha \Theta(\{\mathbf{a}_j\}, \{\rho_j\}). \quad (50)$$

The states X and Y are characterized by the instanton number K combined with the set of coordinates $\{\mathbf{a}_j, \rho_j\}$.

In canonical algorithms $\omega(X, Y)$ is usually chosen to be symmetric so that it is omitted in (49) without further comments. This is not possible in a grand canonical ensemble, where one has to consider transitions that

change the instanton number. Independent of the choice of $\omega(X, Y)$ there will be additional volume factors in a_{XY} for processes that do not conserve the instanton number. This results from the asymmetry in particle creation and destruction. If an instanton is created one has to specify a probability for the generation of its new coordinates, On the other hand, in the process of removing an instanton such a probability does not occur. In our case, for the space-coordinates as well as for the radii we choose a uniform distribution within the allowed volume.

In the algorithm three different kinds of steps occur with equal probability: creation, destruction and movement of an instanton. With the shortcut notation

$$\star = \begin{cases} 1. & \|\mathbf{a}_i - \mathbf{a}_j\| > \left(\frac{\tau}{v_1}\right)^{\frac{1}{d}} (\rho_i + \rho_j) \quad \forall i, j, \text{ and} \\ 2. & |a_i^\mu| < \frac{1}{2} V^{\frac{1}{d}} \quad \forall \mu, i, \text{ and} \\ 3. & 0 < \rho_i < \frac{1}{2} \left(\frac{v_1}{\tau} V\right)^{\frac{1}{d}} \quad \forall i \end{cases}$$

we have chosen the following transition rules, where x denotes a random number between 0 and 1.

– Creation:

The creation of a new instanton with number $K + 1$ and coordinates (\mathbf{a}', ρ') is proposed, and

$$X \rightarrow \begin{cases} Y, & \frac{CV(Vv_1/\tau)^{\frac{1}{d}}}{2(K+1)} \rho'^\alpha \geq x \text{ and } \star \\ X, & \frac{CV(Vv_1/\tau)^{\frac{1}{d}}}{2(K+1)} \rho'^\alpha < x \text{ or not } \star. \end{cases}$$

– Destruction:

The destruction of an instanton with randomly chosen number j is proposed, and

$$X \rightarrow \begin{cases} Y, & \frac{2K}{CV(Vv_1/\tau)^{\frac{1}{d}}} \rho_j^{-\alpha} \geq x \\ X, & \frac{2K}{CV(Vv_1/\tau)^{\frac{1}{d}}} \rho_j^{-\alpha} < x. \end{cases}$$

– Movement:

A movement of a randomly chosen instanton in a volume element $[-\delta_a, \delta_a]^d \times [-\delta_\rho, \delta_\rho]$ around the original coordinates is proposed, and

$$X \rightarrow \begin{cases} Y, & \left(\frac{\rho'_j}{\rho_j}\right)^\alpha \geq x \text{ and } \star \\ X, & \left(\frac{\rho'_j}{\rho_j}\right)^\alpha < x \text{ or not } \star. \end{cases}$$

The simulations were started with the empty configuration ($K = 0$). Measuring was started after the instanton number reached saturation.

4.2 Simulation results

In the case of $d = 1$ dimensions the available exact result (32) provides a useful check of the Monte Carlo calculations. In Fig. 2 Monte Carlo data for $n(\rho)$ in $d = 1$ are compared with the exact formula. The size L has been chosen large enough such that finite L effects are negligible. Obviously the Monte Carlo data agree very well with

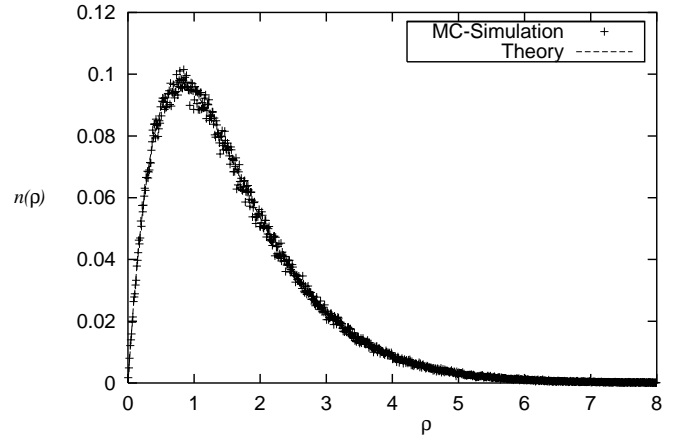


Fig. 2. The size distribution $n(\rho)$ in $d = 1$ from a Monte Carlo simulation with $C = 1$, $\tau = 2$, $\alpha = 1$, $V = L = 2000$ in comparison with the predictions of formula (32)

the theoretical predictions, and the thermodynamic limit has been approached sufficiently.

With this check on the Monte Carlo algorithm we proceed to the more interesting case of four space-time dimensions ($d = 4$). In order to compare the Monte Carlo data with the outcome of our analytical approximations, (46), we have to make assumptions concerning the parameter $h(d) = h(4)$ that describes the ability of instantons to fill a given volume. We consider three choices, namely the lower bound $h_1 = 1$, the upper bound $h_2 = 2^{4-1} = 8$ and their geometric mean $h_g = 2\sqrt{2}$. The parameter τ is taken to be equal to v_1 . For the volume we chose $V = 15^4$. This is based on simulations in different volumes, which showed that in this case the thermodynamic limit was approximately reached within the errors of the simulation. The parameter α is taken to be $\alpha = 7/3$, which is the value for SU(2) gauge theory in 4 dimensions.

Figure 3 shows the Monte Carlo data in comparison with the analytical approximation. Near the maximum of the distribution the approximation qualitatively reproduces the Monte Carlo results. Furthermore, the growth of the distribution for small instanton radii according to a power law with exponent α can be confirmed, as is shown in Fig. 4.

In order to study the behaviour of $n(\rho)$ for large ρ we considered the ratio

$$F(\rho) = \frac{n(\rho)}{\rho^\alpha}. \quad (51)$$

Inspired by the theoretical results we tried fits of the form

$$F_{fit}(\rho) = a \exp(-c\rho^p). \quad (52)$$

The parameter a was obtained by extrapolating $F(\rho)$ to small ρ . The fit with parameters c and p was then obtained using the Marquardt–Levenberg-algorithm. We performed fits for various choices of the model parameters C and α . In agreement with the theoretical results they showed that c depends on α , while p is nearly independent of it.

The main interest is in the exponent p . We present the results for the parameter set $C = 1$, $\alpha = 7/3$, $V = 15^4$,

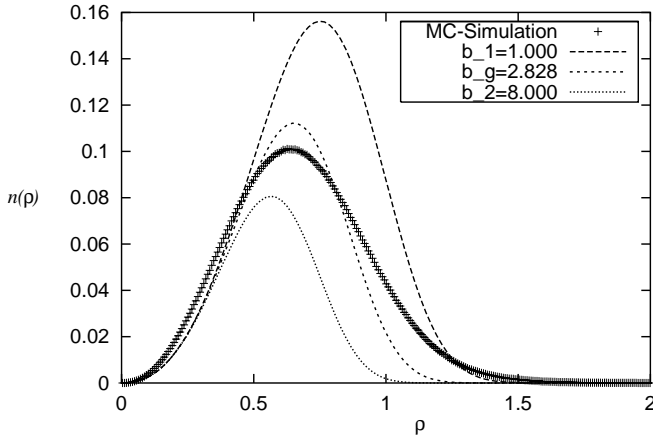


Fig. 3. The size distribution $n(\rho)$ in $d = 4$ dimensions from a Monte Carlo simulation with $C = 1$, $\alpha = 7/3$ ($SU(2)$), $V = 15^4$ in comparison with the predictions of formula (46) for $h(4) = 1$, $2\sqrt{2}$, and 8

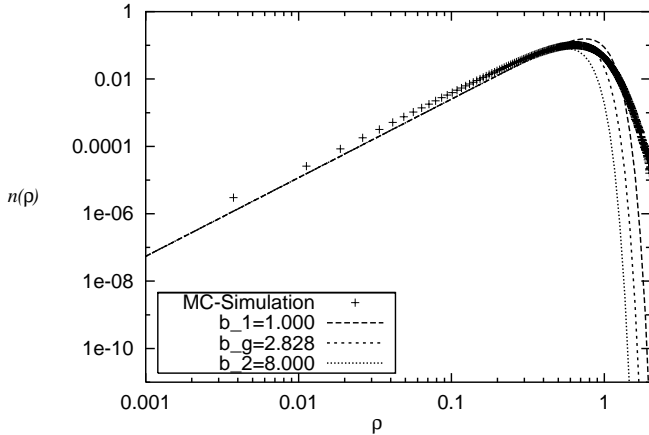


Fig. 4. The size distribution $n(\rho)$ in $d = 4$ dimensions from a Monte Carlo simulation with $C = 1$, $\alpha = 7/3$ ($SU(2)$), $V = 15^4$ in comparison with the predictions of formula (46) for $h(4) = 1$, $2\sqrt{2}$, and 8, plotted on a double logarithmic scale

because this value of α is relevant for gauge theory with gauge group $SU(2)$. For $\alpha = 6$, the $SU(3)$ case, the results for p are the same within the present errors. We find $a \approx 0.89$, and the fit leads to $c = 3.3 \pm 0.2$ and $p = 1.9 \pm 0.2$. In Fig. 5 the result of a fit in the interval $[0, 2.25]$ is shown.

This Gaussian ($p = 2$) decay of the probability distribution has already been predicted by some authors under various assumptions [11,18]. A recent approach based on an idea of dual superconductivity [19] also leads to the prediction $p = 2$. Furthermore good agreement with the $SU(3)$ lattice gauge theory calculations of Hasenfratz et al. [17] was found.

The exponent $p = 2$ differs from the one predicted by our approximate analytical calculation, $p = d$. The saddle point approximation being made is, however, not beyond any doubt. In that case the exponent originates from the effective excluded volume being proportional to ρ^d at the considered saddle point. This would also coincide with the intuitive expectation based on the following

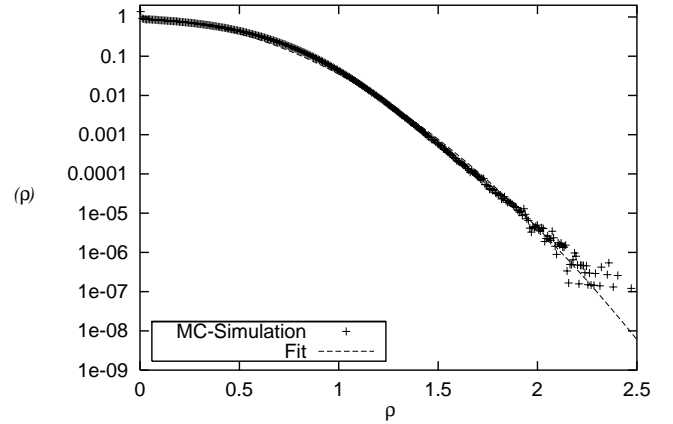


Fig. 5. $F(\rho) = n(\rho)/\rho^\alpha$ in $d = 4$ dimensions from a Monte Carlo simulation with $C = 1$, $\alpha = 7/3$ ($SU(2)$), $V = 15^4$ in comparison with the fit $F_{fit}(\rho)$ with $a = 0.89$, $c = 3.24$, $p = 1.92$, plotted on a logarithmic scale

picture. In the presence of a large instanton of size ρ the remaining ones are excluded from a volume $\sim \rho^d$. If they behave like a dilute gas, one would expect that the excluded volume yields a suppression factor $\propto \exp(-c\rho^d)$. The instanton ensemble in the effective remaining volume is, however, dominated by dense configurations, as the analytical calculation shows. Therefore the intuitive picture should be considered with reservations. Indeed, the calculations of [11] take into account excluded volume effects in the framework of the theory of grand canonical pair distribution functions and, also employing certain approximations, arrive at $p = 2$.

In recent years much effort has been devoted to lattice Monte Carlo calculations of properties of the instanton ensemble. There are still ambiguities due to smoothing procedures and only data with little statistics are yet available. Nevertheless some quantitative statements have been given. Concerning the size distribution for small ρ , lattice calculations appear to support the power law (7) rather than (9). A nice plot, using data of [16], can be found in [30]. For the large- ρ distribution, de Forcrand et al. predict an exponential decrease with $p = 3 \pm 1$ from their $SU(2)$ lattice data [15]. In contrast to this, Smith and Teper conclude from their $SU(3)$ simulations a decay according to $\rho^{-\xi}$ with $\xi \approx 10 \dots 12$ [16].

5 Conclusion

We have studied the distribution of instanton sizes ρ in the framework of a model, where instanton interactions are approximated by a hard core potential with variable radius. This model incorporates the basic features of a dynamical cut-off on large instanton sizes. In the one-dimensional case an exact formula can be derived, which yields a power-like growth $\sim \rho^\alpha$ for small radii ρ and an exponential decay for large ρ .

In four space-time dimensions we employed analytical approximations as well as Monte Carlo simulations. The

theoretical calculations generalize the one-dimensional results and give a power-like behaviour for small ρ . For large radii ρ they overestimate the decay which is found in the Monte Carlo data. Fits to the numerical Monte Carlo results suggest a behaviour like

$$n(\rho) \stackrel{\rho \rightarrow \infty}{\sim} \exp(-c\rho^2), \quad (53)$$

in agreement with some other work on gauge theories.

The results indicate that our simplified model reproduces the main features of instanton ensembles with a dynamical infrared cut-off. Definite results about properties of instanton ensembles can of course only be expected from future Monte Carlo calculations of lattice gauge theories.

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